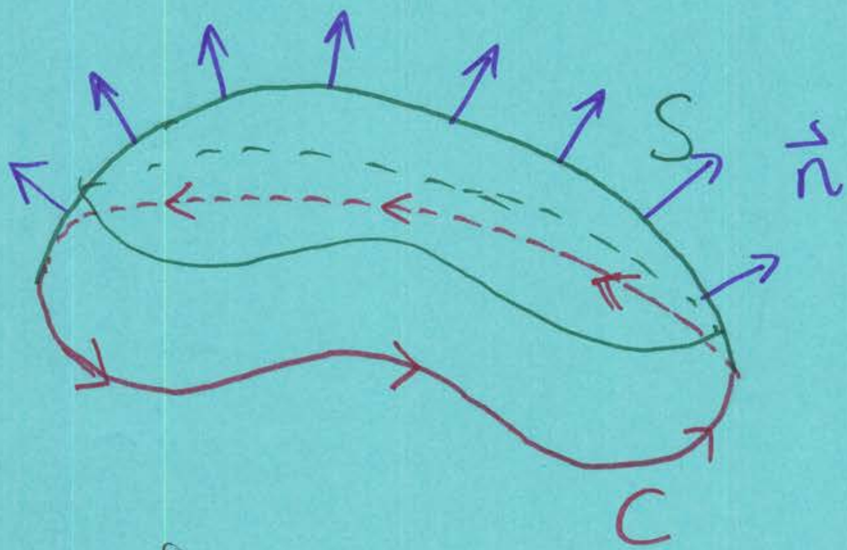
Lecture 42

16.8 - Stokes' Theorem.

Let S be a surface with boundary C. and orientation $\vec{n}$.

E.g.:



We say that <u>C has orientation consistent with S</u> if while you traverse the curve with your head in the direction of $\vec{n}$, the surface is on your left. In this sense, we say S induces an orientation on C.

Stokes' Theorem

Let S be an oriented, piecewise-smooth surface which is bounded by a simple, closed, piecewise smooth curve C, and give C the orientation induced by S. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ which is C^1 in an open region containing S. Then

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot d\vec{S}}$$

Let us now revisit the problem at the end of Wednesday's lecture notes:

Alt. Sol: This is a problem to which Stokes' theorem applies. C is the circle $x^2+y^2=4$, $z=1$ (obtained by plugging $z=1$ into $z=5-x^2-y^2$). The orientation induced on C is the counterclockwise one, when viewed from above. So, C is parametrized by

$\vec{r}(t)=\langle 2\cos t, 2\sin t, 1\rangle$, $0\le t\le 2\pi$

$\vec{F}(\vec{r}(t)) = \langle -4\sin t, 2\sin t, 6\cos t \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 8\sin^2 t + 4\sin t \cos t$

$$\iint_S (\text{curl } \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (8\sin^2 t + 4\sin t \cos t)\, dt$$

$(\sin^2 t = \frac{1}{2}(1 - \cos 2t))$

$$= \int_0^{2\pi} (4 - 4\cos 2t + 4\sin t \cos t)\, dt = (4t - 2\sin 2t + 2\sin^2 t)\Big|_0^{2\pi}$$

$= 8\pi$

$\diamond$

Notice how much quicker this was!

The other direction is also useful:

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle xy, 2z, 3y \rangle$ and C is the curve of intersection between $x+z=5$ and $x^2+y^2=9$, oriented counterclockwise when viewed from above.

Sol: While it might not be terrible to parametrize C:

$$\vec{r}(t) = \langle 3\cos t, 3\sin t, 5 - 3\cos t \rangle, \quad 0 \le t \le 2\pi,$$

observe:

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle 9\cos t \sin t, 10 - 6\cos t, 9\sin t \rangle \cdot \langle -3\sin t, 3\cos t, 3\sin t \rangle = \underset{\text{JUNK}}{\text{NASTY}}$$

To use Stokes' theorem, we need a surface S which has C as its boundary and induces the correct orientation on C. For the surface itself, we can choose the piece of the plane inside $x^2+y^2=9$, since it's easy to parametrize, so, now it's a matter of orientation. To induce the correct orientation on C, S must have the upward orientation. First, parametrize S: (remember $x+z=5$)

$$\vec{r}(x,y)=\langle x,y,5-x\rangle,\ \{x^2+y^2\le 9\}=D$$

$$\vec{r}_x=\langle 1,0,-1\rangle,\ \vec{r}_y=\langle 0,1,0\rangle.$$

$$\vec{r}_x\times\vec{r}_y=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&0&-1\\0&1&0\end{vmatrix}=\langle 1,0,1\rangle$$ gives the upward orientation. Now, we need the curl of $\vec{F}$:

$$\text{curl }\vec{F}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\xy&2z&3y\end{vmatrix}=\langle 3-2,0,-x\rangle=\langle 1,0,-x\rangle$$

$$(\text{curl }\vec{F})(\vec{r}(x,y))=\langle 1,0,x\rangle,\ (\text{curl }\vec{F})(\vec{r}(x,y))\cdot(\vec{r}_x\times\vec{r}_y)=1+0-x=1-x$$

So, $\int_C\vec{F}\cdot d\vec{r}=\iint_S(\text{curl }\vec{F})\cdot d\vec{S}=\iint_D(1-x)\,dA=\int_0^{2\pi}\int_0^3(1-r\cos\theta)\,r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^3 (r - r^2\cos\theta)\,dr\,d\theta = \int_0^{2\pi} \left(\tfrac{1}{2}r^2 - \tfrac{1}{3}r^3\cos\theta\right)\Big|_0^3 \,d\theta$$

$$= \int_0^{2\pi} \left(\tfrac{9}{2} - 9\cos\theta\right)d\theta = \left(\tfrac{9}{2}\theta - 9\sin\theta\right)\Big|_0^{2\pi} = 9\pi$$

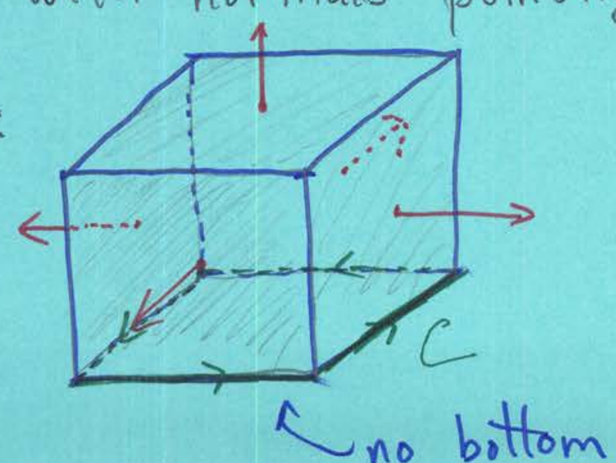
◇

There's one more use of Stokes' theorem worth mentioning: As long as two surfaces S_1 & S_2 have the same boundary C, and both induce the same orientation on C, and all these (S_1, S_2, & C) satisfy Stokes' theorem, then

$$\iint_{S_1} (\text{curl } \vec{F})\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \iint_{S_2} (\text{curl } \vec{F})\cdot d\vec{S}$$

As an example, let S_1 be the surface of the box $[0,1]\times[0,1]\times[0,1]$, without the bottom. Orient S_1 with normals pointing away from the box:

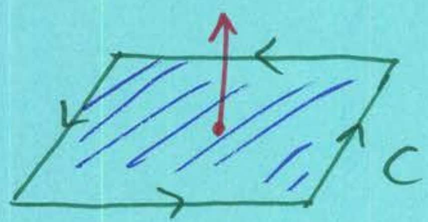
S_1:



This would be frustrating to integrate over (5 surface integrals!), and even its boundary wouldn't be fun to do a line integral over (4 of them...)

However, we can replace S_1 by just the bottom of the box, i.e. the square $[0,1]\times[0,1]\times\{0\}$, oriented upward:

S_2:



<u>Much</u> easier!